EFFECT OF ENTANGLEMENT ON COOPERATIVE GAMES

An Open-Ended Lab Project Report Submitted
in Partial Fulfillment of the Requirements
for the Degree of

Bachelor of Technology

by

Pranav Guruprasad Rao
(112101038)

IIT PALAKKAD

COMPUTER SCIENCE AND ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY PALAKKAD

Department of Computer Science and Engineering

INDIAN INSTITUTE OF TECHNOLOGY PALAKKAD

Certificate

This is to certify that the work contained in this report entitled "**Effect of Entanglement in Cooperative Games**" is a bonafide work of **Pranav Guruprasad Rao** (**Roll No. 112101038**), carried out in the Department of Computer Science and Engineering, Indian Institute of Technology Palakkad under my supervision and it has not been submitted elsewhere.

Dr Krishnamoorthy Dinesh & Dr Srimanta Bhattarcharya
(Project Guides)

Date: 6th December 2023

Contents

Chapter 1

Introduction

In this project, we shall be looking at Cooperative Games where players can collaborate to win the game.
We are interested in the games where a Strategy involving an Entangled pair performs better than any classical strategy.

1.1 Non-Local Games

Non-local Games refer to the Games where the players are separated and their play doesn't affect the other.

Setup of the Game:
There are three players, Alice (A), Bob (B) and Referee (R).
The Referee picks $(s, t) \sim \mu$ (a given distribution from a finite set $S \times T$) and sends s to Alice and t to Bob. Alice and Bob cannot communicate with each other after the game has started.
Alice and Bob send a message $a(s)$ and $b(t)$ ($a : S \to A$, $b : T \to B$), respectively to the Referee who decides if they won or lost based on a verification criteria V.
Alice and Bob have the knowledge of μ and the V and are allowed to agree on a strategy before the start of the game.

1.2 Classical and Quantum Values of Games

Given a Game $G(V, \pi)$ where V,π are as defined above, we would like to find the winning probability for a given classical or quantum strategy.

The Classical Value for the Game G is the maximum probability with which Alice and Bob can win G, ranging over all purely classical strategies, given by

$$\omega_c(G(V, \pi)) = \max_{a,b} \sum_{s,t} \pi(s, t) V(a(s), b(t)|s, t)$$

The Quantum Value for the Game G is the supremum of the winning probabilities over all quantum strategies of Alice and Bob given by

$$\omega_q(G(V,\pi)) = \sum_{(s,t)\in S\times T} \pi(s,t) \sum_{(a,b)\in A\times B} \langle\psi X_b^a \otimes Y_t^b|\psi\rangle V(a,b|s,t)$$

1.3 Qubit

A Qubit is a two-level Quantum Mechanical System which can be represented as a 2 dimension Complex vector as:

$$|\psi\rangle = a|0\rangle + b|1\rangle; |a|^2 + |b|^2 = 1, a, b \in \mathbb{C}$$

a & b are called amplitudes. Any operation on a qubit can be represented by Unitary matrices.

Measurement of a qubit is the collapsing of its state into a single state which is the final observation. This is done by a set of Measurement operators for each corresponding collapsed state. A Measurement is probabilistic, and we have probabilities for each outcome.

E.g -
When we measure $|\psi\rangle$ as given above, we get -

- 0 with probability $|a|^2$
- 1 with probability $|b|^2$

A collection of qubits $\{q_1, q_2, ..., q_k\}$ can be collectively represented by a combined state $|q_1q_2...q_k\rangle$.

1.4 Unitary Operations

The Rotation Matrices in standard basis $\{|0\rangle,|1\rangle\}$ are of the form:

$$U_a = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}, a \in \mathbf{R} \tag{1.1}$$

U_a is a Change of Basis Matrix, which changes the basis from $\{|0\rangle,|1\rangle\}$ to $\{|\psi_a\rangle,|\psi_{a+\pi/2}\rangle\}$ where

$$|\psi_a\rangle = \cos a|0\rangle + \sin a|1\rangle \tag{1.2}$$

Each Rotation corresponds to a measurement on a certain basis in Quantum Mechanics.

Based on this from Eq. 1.1,

$$U_a = |0\rangle\langle\psi_a| + |1\rangle\langle\psi_{a+\pi/2}| \tag{1.3}$$

Inner product (on a standard basis) of

$$\langle\psi_a|\psi_b\rangle = \cos a \cos b + \sin a \sin b = \cos(a-b) \tag{1.4}$$

Action of Unitary Operation U_a on $|\psi_b\rangle$:

$$U_a|\psi_b\rangle = |\psi_{b-a}\rangle$$

1.5 Entanglement

A pair of qubits is said to be entangled if the combined state of them cannot be separated into independent qubit states. i.e $|q_1, q_2\rangle \neq |q_1\rangle \otimes |q_2\rangle$.

Another interesting property is that measurement of one qubit completely specifies the state of the other qubit. e.g.

Measurement of 1st qubit of $|q_1, q_2\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ yields two possibilites:

- Measurement 0, with $|q_1\rangle = |0\rangle$
- Measurement 1, with $|q_1\rangle = |1\rangle$

1.5.1 Entangled States

We shall be focusing on two-qubit entangled pairs.

1.5.2 Bell Basis

There are 4 Standard Two-Qubit Entangled states known as Bell States. These states are maximally entangled (the measurement of one qubit completely specifies the measurement of the other).
These states are :

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, |\phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, |\psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, |\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{1.5}$$

We can then conclude that

$$|00\rangle = \frac{|\phi^+\rangle + |\phi^-\rangle}{\sqrt{2}}, |11\rangle = \frac{|\phi^+\rangle - |\phi^-\rangle}{\sqrt{2}}, \tag{1.6}$$

$$|01\rangle = \frac{|\psi^+\rangle + |\psi^-\rangle}{\sqrt{2}}, |10\rangle = \frac{|\psi^+\rangle - |\psi^-\rangle}{\sqrt{2}} \tag{1.7}$$

<u>Claim:</u> The set $\{|\phi^+\rangle,|\phi^-\rangle,|\psi^+\rangle,|\psi^-\rangle\}$ form a basis to $\mathbf{C}^2 \otimes \mathbf{C}^2$
<u>Proof:</u>
Standard Basis of $\mathbf{C}^2 = |0\rangle, |1\rangle$
Standard Basis of $\mathbf{C}^2 \otimes \mathbf{C}^2 = |00\rangle, |11\rangle$
let $|\psi\rangle \in \mathbf{C}^2 \otimes \mathbf{C}^2$

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle; \alpha, \beta, \gamma, \delta \in \mathbf{C}$$

Thus, from eqs. (1.6) and (1.7)

$$|\psi\rangle = \alpha\frac{|\phi^+\rangle + |\phi^-\rangle}{\sqrt{2}} + \beta\frac{|\psi^+\rangle + |\psi^-\rangle}{\sqrt{2}} + \gamma\frac{|\psi^+\rangle - |\psi^-\rangle}{\sqrt{2}} + \delta\frac{|\phi^+\rangle - |\phi^-\rangle}{\sqrt{2}}$$

$$|\psi\rangle = \frac{\alpha+\delta}{\sqrt{2}}|\phi^+\rangle + \frac{\beta+\gamma}{\sqrt{2}}|\psi^+\rangle + \frac{\beta-\gamma}{\sqrt{2}}|\psi^-\rangle + \frac{\alpha-\delta}{\sqrt{2}}|\phi^-\rangle \tag{1.8}$$

$$\frac{\alpha+\delta}{\sqrt{2}}, \frac{\beta+\gamma}{\sqrt{2}}, \frac{\beta-\gamma}{\sqrt{2}}, \frac{\alpha-\delta}{\sqrt{2}} \in \mathbf{C}$$

■

1.6 4 Dimensional Rotations on Bell States

Given any $|\psi_a\rangle$ & $|\psi_b\rangle \in \mathbf{C}^2, a, b \in \mathbf{R}$, using Eq. 1.4

$$\langle \psi_a \otimes \psi_b|\phi^+\rangle = \frac{\cos(a-b)}{\sqrt{2}} = \frac{\langle\psi_a|\psi_b\rangle}{\sqrt{2}}$$

We perform 2-dimensional Rotations on each qubit of a bell pair.

$$(U_a \otimes U_b)|\phi^+\rangle = \cos(a-b)|\phi^+\rangle + \sin(a-b)|\psi^-\rangle \tag{1.9}$$

$$(U_a \otimes U_b)|\phi^-\rangle = \cos(a+b)|\phi^-\rangle - \sin(a+b)|\psi^+\rangle \tag{1.10}$$

$$(U_a \otimes U_b)|\psi^+\rangle = \cos(a+b)|\psi^+\rangle + \sin(a+b)|\phi^-\rangle \tag{1.11}$$

$$(U_a \otimes U_b)|\psi^-\rangle = \cos(a-b)|\psi^-\rangle - \sin(a-b)|\phi^+\rangle \tag{1.12}$$

From the eqs. (1.9) to (1.12) we can see a relation between $|\phi^+\rangle$ & $|\psi^-\rangle$ as well as between $|\phi^-\rangle$ & $|\psi^+\rangle$ given as, $\forall a, b \in \mathbf{R}$

$$|\psi^-\rangle = [\cot(a-b)\mathbf{I} - \csc(a-b)(U_a^{-1} \otimes U_b^{-1})]|\phi^+\rangle \tag{1.13}$$

$$|\psi^+\rangle = -[\cot(a+b)\mathbf{I} - \csc(a+b)(U_a^{-1} \otimes U_b^{-1})]|\phi^-\rangle \tag{1.14}$$

1.7 Bell Theorem

Our project aims to seek examples where quantum strategies perform better than any classical strategy, using entanglement. This fact is encapsulated by the famous Bell's Thereom:

Theorem 1 *(Bell's Theorem): No theory of local realism such as a local hidden variables theory can account for the correlations between entangled electrons predicted by quantum mechanics.*

1.8 General Quantum Strategy

Our Quantum Strategy will follow the following schema:

1. We will define a separate Unitary Operation for each vertex question.

2. By the application of these operations on the entangled pair $|\phi^+\rangle$, we note the probabilities of each output in a Winning Probability table.

3. We write down the Optimization Problem.

4. We equate one angle to be zero (signifying standard basis measurement)

5. We try to reduce the variables in the equation using properties, and finally try to derive cases which beat the classical probability.

Chapter 2

CHSH

CHSH stands for the game's creators' names: John Clauser, Michael Horne, Abner Shimony, Richard Holt

In this game $G_{CHSH}(V, \pi)$ we have,

- $S = T = A = B = \{0, 1\}$
- $V(a, b|s, t) : (s \wedge t) = (a \oplus b)$
- $\pi(s, t)$ to be uniform distribution over $S \times T$

We shall look at strategies to win this game. We are interested in the Quantum strategy used in the CHSH Game and its winning probability.

2.1 Classical Strategy

There are a total of four possible deterministic strategies:

1. Constant 0: $a = b = 0\ \forall s, t \in \{0, 1\}$
2. Constant 1: $a = b = 0\ \forall s, t \in \{0, 1\}$
3. Identity: $a = s, b = t\ \forall s, t \in \{0, 1\}$
4. Complementary: $a = \bar{b} \forall s, t \in \{0, 1\}$

We compare the four strategies according to the 4.1 and obtain as:
Optimal Classical Strategy: Answer either a 0 or a 1 for all questions asked.
Thus, $\omega_c = 75\%$.

A probabilistic strategy cannot do better than deterministic, as they are composed of deterministic strategies.

s	t	$s \wedge t$	Winning Condition of (a, b)
0	0	0	$a = b$
0	1	0	$a = b$
1	0	0	$a = b$
1	1	1	$a \neq b$

Table 2.1: Winning Condition Table

2.2 Quantum Strategy

Here is the complete description of a general Quantum Strategy:

1. Alice and Bob share an entangled pair of qubits of state $|\psi\rangle$ (one qubit for each)
2. Either of them can perform Unitary operations on their respective qubit and measure on some basis.
3. After measuring their qubit, each sends the output to the Referee, who then verifies the predicate whether they won or lost

Let us see the case when they share a pair in Bell State of $|\phi^+\rangle$ defined in eq. (1.5)

Unitary operations by Alice

- U_{a_0} when $s = 0$
- U_{a_1} when $s = 1$

Unitary operations by Bob

- U_{b_0} when $s = 0$
- U_{b_1} when $s = 1$

where $a_0, a_1, b_0, b_1 \in \mathbf{R}$

There are a total of 4 unknowns.

s	t	$s \vee t$	Accepted Answers	Probabilities of these Answers
0	0	0	{(00),(11)}	$\cos^2(a_0 - b_0)$
0	1	0	{(00),(11)}	$\cos^2(a_0 - b_1)$
1	0	0	{(00),(11)}	$\cos^2(a_1 - b_0)$
1	1	1	{(01),(10)}	$\sin^2(a_1 - b_1)$

Table 2.2: Winning Probabilities of the Strategy

Now we need to maximise the minimum of these probabilities such that

$$\max_{a_0,a_1,b_0,b_1} (\min(\cos^2(a_0 - b_0), \cos^2(a_0 - b_1), \cos^2(a_1 - b_0), \sin^2(a_1 - b_1))) > \frac{3}{4} \tag{2.1}$$

WLOG lets us fix a certain angle as 0 (Fixing a certain measurement on a standard basis) Let $a_0 = 0$, then from (2.1)

$$\max_{a_1,b_0,b_1} (\min(\cos^2(b_0), \cos^2(b_1), \cos^2(a_1 - b_0), \sin^2(a_1 - b_1))) > \frac{3}{4} \tag{2.2}$$

2.2.1 Better Quantum Strategies

Let us first show that $\exists\ a_1, b_0, b_1 \in \mathbf{R}$ which satisfies the inequality. We look for the values of a_1, b_0, b_1 for which the inequality (2.2) holds with the Figure 2.1

Constraints on a_1, b_0, b_1:

1. $\frac{-\pi}{6} < a_0, b_1, (a_1 - b_0) < \frac{\pi}{6}$
2. $\frac{\pi}{3} < (a_1 - b_1) < \frac{2\pi}{3}$
3. $a_0 \neq a_1$
4. $b_0 \neq b_1$

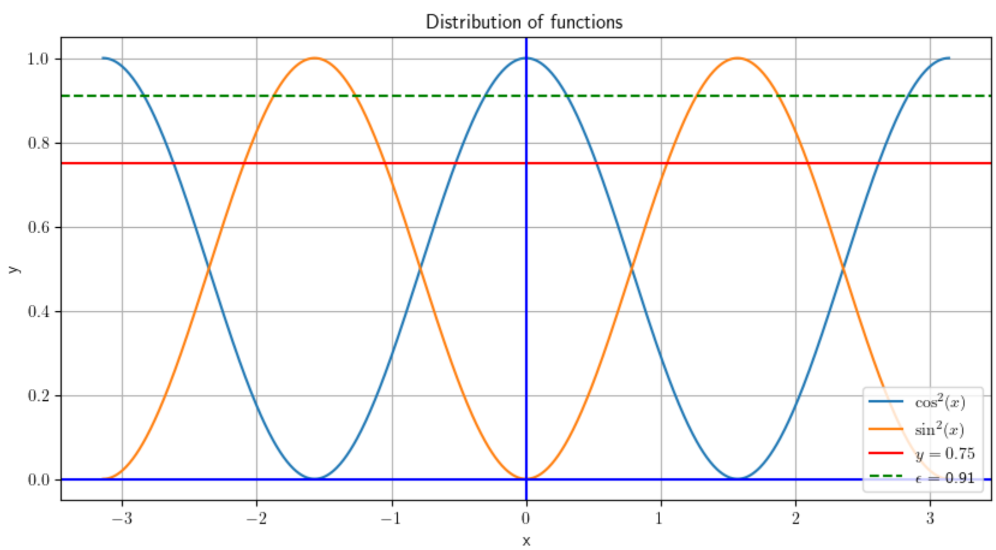


Figure 2.1: Cosine Sine graph

Constraints 1 & 2 are obtained from Figure (2.1).

Constraints 3 & 4 are obtained from Proof by Contradiction:

Proof: Assume Constraint 4 to be false i.e. $b_0 = b_1$. Then the inequality (2.2) simplifies to

$$\max_{a_1,b_1}(\min(\cos^2 b_1), \cos^2(a_1 - b_1), \sin^2(a_1 - b_1))) > \frac{3}{4}$$

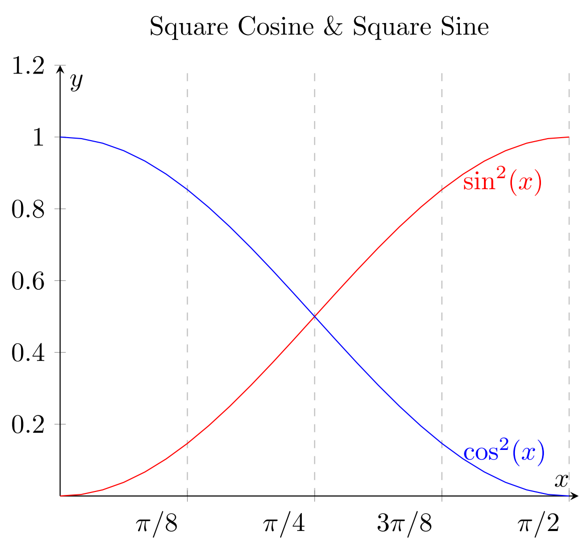
From Figure (2.1), the only point where $\cos^2(x)$ graph intersects $\sin^2(x)$ is at $\frac{\pi}{4}$.
But $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{2} < \frac{3}{4}$ (violates inequality (2.2)
(Contradiction) ■
Similar reasoning can be made for Constraint 3.

WLOG let $b_1 > b_0$ (satisfies Constraint 4)
Expressing a_1 wrt b_0, b_1 wrt a_1, we get

$$\frac{-\pi}{6} < b_0 < b_1 < \frac{\pi}{6}$$
$$b_0 - \frac{\pi}{6} < a_1 < b_0 + \frac{\pi}{6}$$
$$a_1 - \frac{2\pi}{3} < b_1 < a_1 - \frac{\pi}{3}$$

As b_0 & b_1 are over a range, we see for each of these values of a_1, b_0, b_1 we get a winning probability $> \frac{3}{4}$
Thus, There exists a Quantum Strategy which beats the best classical strategies.

2.2.2 Optimal Quantum Strategy



We begin by noticing the following behaviour of Cosine and Sine functions.

Cosine function is a decreasing function from $[0, \frac{\pi}{2}]$
Sine function is an increasing function from $[0, \frac{\pi}{2}]$

Thus coupled together, they must be maximised when they are equal, i.e

$$\max_x(\min(\cos^2(x), \sin^2(x))) : x = \frac{\pi}{4}$$

We can use this fact in solving the CHSH equation.
Notice that $cos^2(a_1 - b_0)$ & $sin^2(a_1 - b_1)$ have the common angle a_1. So $a_1 = \frac{\pi}{4}$ for both to be equally maximum.

We know that $b_0 \neq b_1$ and WLOG we take $b_0 < b_1$.
So our equation becomes

$$\max_{b_1}(\min(\cos^2(b_1), \cos^2(\frac{\pi}{4} - b_0), \sin^2(\frac{\pi}{4} - b_1)))$$

Now, we have the cosine term $cos^2(\frac{\pi}{4} - b_0)$ lesser than $sin^2(\frac{\pi}{4} - b_1)$. To satisfy the condition, both would be on opposite sides of the line $x = \frac{\pi}{4}$, and maximality would be achieved when $b_0 = -b_1$

This reduces the equation to:

$$\max_{b_0}(\min(\cos^2(b_0), \cos^2(\frac{\pi}{4} - b_0)))$$

This can now be easily verified by equating both sine and cosine terms to get the optimum of $b_0 = -\pi/8$. So We get the optimum values as:

$$a_0 = 0, a_1 = \frac{\pi}{4}, b_0 = \frac{-\pi}{8}, b_1 = \frac{\pi}{8}$$

The winning probability comes as:

$$\omega_q = \cos^2(\frac{\pi}{8}) = 85.3\%$$

2.2.3 Corollary 1

We shall see for the other Bell States

- $|\psi^-\rangle$:
  By equation (1.13), We establish that there exists a Unitary transformation from $|\phi^+\rangle$ to $|\psi^-\rangle$
  Hence the winning probability remains the same (85.3%)
- $|\phi^-\rangle$:
  By equation (1.14), We substitute ψ by $-\psi$
  The rest follows the same as above. Hence the winning probability remains the same (85.3%)
- $|\psi^+\rangle$:
  By Case 1 & 2, we can establish the winning probability as (85.3%)

2.2.4 Corollary 2

We shall see for any general two-qubit Entangled states.
From eq. (1.8), we showed that a two-qubit entangled state can be uniquely represented by the Bell Basis. Thus from linearity, it follows that the winning probability of this strategy is 85.3% as well.

Chapter 3

Odd Cycle Game

Alice and Bob have with them an N-vertex Odd Cycle, and they have to convince the Referee that the cycle is not 2-colorable i.e. at least one adjacent vertex-pair has the same color.

Game's Formal definition: (for $n \in \mathbf{N}$ where n is even)

1. Question Sets : $S = T = \mathbf{Z}_n$
2. Question distribution : π over $\{(s,t) \in \mathbf{Z}_n \times \mathbf{Z}_n : s = t || s+1 = t (\text{mod n})\}$
3. Answer Sets : $A = B = \{0,1\}$
4. Predicate:

$$V(a,b|s,t) = \begin{cases} 1, & \text{if } a \oplus b = [s+1 \equiv t \text{ mod n}] \\ 0, & \text{otherwise} \end{cases}$$

<u>Claim:</u> Odd Cycles are not 2-colorable

<u>Proof:</u> Contrapositive Statement: No 2-colorable graphs are Odd Cycles
Proof of Contrapositive:
Let $G = (V,E)$ be a 2-colorable graph and C be the coloring function defined as:

$$C : V \to \{0,1\}$$

WLOG lets us assume G is a connected graph.
Thus C partitions V into A & B ($A \cup B = V$).
WLOG assume no vertex in either A or B share an edge.
Thus

$$\forall (x,y) \in E, x \in A, y \in B$$

Thus, G is a bipartite graph which cannot be odd.
Hence Proved. ■

3.1 Classical Strategy

There exists no perfect classical strategy ($\omega_c = 1$). We can thus hope to find a case with maximum ω_c.

By colouring the graph in two colours, we are essentially (almost) bipartiting the graph into two partitions.
Given a color function $C : \mathbf{Z}_n \rightarrow \{0, 1\}$ Let the partitions be defined as:

$$V = \{x | x \in \mathbf{Z}_n, C(x) = 0\}$$

$$B = \{x | x \in \mathbf{Z}_n, C(x) = 1\}$$

$$V \cap B = \phi$$

As Vertex Set is odd-sized, one of the partitions' size is greater than the other.
For any coloring scheme, we must maximise the no of adjacent vertices colored differently according to the given predicate.
So we colour wrt the following algorithm:

1. Start with sets $A = B = \phi$ and coloring function C.
2. Pick a vertex $v \in V$ in random.
   (a) Add v to A.
   (b) $C(v) = 0$.
3. For i such that $(v, i) \in E$ & $i \notin A \& \notin B$:
   (a) Add i to B.
   (b) $C(i) = 1$.
4. Pick a random i from above.
5. Repeat till all vertices are coloured.

This can be successfully done using the coloring map:

$$C(i) = i\%2$$

Thus, for a given vertex, Alice and Bob need to respond the modulo 2 of the question.
However, we will have one adjacent pair which would be colored the same. There are a total of $2n$ pairs.
Thus the winning probability comes as

$$\omega_c = 1 - \frac{1}{2n}$$

3.2 Quantum Strategy

The General Quantum Strategy is modified as: So Unitary Operations are :

$$U_A = \{U_{a_i} | i \in S, a_i \in [0, \pi]\}$$

$$U_B = \{U_{b_i} | i \in T, b_i \in [0, \pi]\}$$

Thus the Winning Probability Table will be as follows:

s	t	Accepted Answers	Probabilities of these Answers
i	i	{(0,0),(1,1)}	$\cos^2(a_i - b_i)$
i+1	i	{(0,1),(1,0)}	$\sin^2(a_{i+1} - b_i)$

Table 3.1: Winning Probability Table

We thus obtain the optimization equation as:

$$\max_{a_i, b_i, a_{i+1}} (\min(\cos^2(a_i - b_i), \sin^2(a_{i+1} - b_i)))$$

First, we can always fix one angle to be 0. Let $a_0 = 0$

Claim: $a_i \neq a_j \forall i, j \in \mathbf{Z}_n$

Proof: Proof by Contradiction:
Let $a_i = a_j \forall i, j \in \mathbf{Z}_n$
Then, there exists a pair in the equation as $(cos^2(a_i - b_j), sin^2(a_i - b_j))$ which we know to be maximised only at $a_i - b_j = \frac{\pi}{4}$ leading to subpar maximum.
A similar argument can be provided for the $b_i = b_j$ case. ■

3.2.1 Similarity to CHSH Game

As we are using the same strategic principle and predicate as CHSH, we would like to see if there is any similarity between the 2 games.

Note the following:
In the case of n=3,
We had made $a_0 = 0$.
We decide, since there is no constraints on $b_0 \neq 0$, we equate it to 0 too.
We then take two angles
$z = \pi/2 - a_1 \& m = \frac{\pi}{2} - b_2, a_2 = \frac{\pi}{2} - b_1$

The equation results in:

$$\max(\min(cos^2(z), cos^2(m), sin^2(b_1 - z), cos^2(b_1 - m)))$$

This equation is the same as that of the CHSH game.
Here we have fixed some angles as same, even though there was bearing to do

so. Hence, this solution can only be at most the maxima.

Hence we have the winning probability of the CHSH game (85.3%) which is greater than ω_c for the Odd cycle with n=3 (83.3%).
So we have a Quantum strategy which performs better than any classical strategy as follows:
$a_0 = 0, a_1 = 3\pi/8, a_2 = \pi/4, b_0 = 0, b_1 = \pi/4, b_2 = 5\pi/8$

Here is another approach which converges towards the CHSH Game:
$a_i = b_i, \forall i \in \mathbf{Z}_n$

3.3 Approaches towards the optima

We shall search for a strategy with a better winning probability.
Let us arrange the probability values in the equation in a table.

$cos^2(a_0 - b_0)$	$sin^2(a_1 - b_0)$
$cos^2(a_1 - b_1)$	$sin^2(a_2 - b_1)$
...	...
$cos^2(a_{n-2} - b_{n-2})$	$sin^2(a_{n-1} - b_{n-2})$
$cos^2(a_{n-1} - b_{n-1})$	$sin^2(a_0 - b_{n-1})$

Table 3.2: Rearangement of Terms

We are to maximise each of these values.
The cosine function is maximised when its angles are minimised and the Sine function is maximised when its angles are maximised (for angles in interval $[0, \pi/2]$)

Thus, for maximising an entry of the form $cos^2(a_i - b_i), sin^2(a_{i+1} - b_i)$, we would want the following relation between angles a_i, b_i, a_{i+1}:
b_i should be as close as possible to a_i and as far from b_{i+1} as possible.

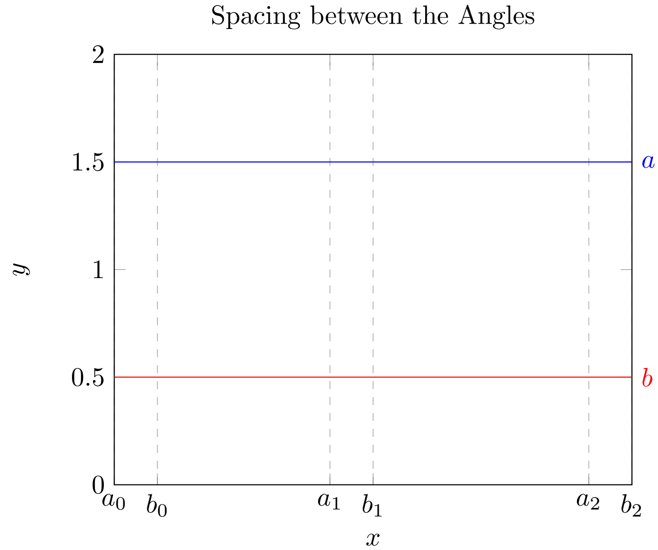
Accounting for this for all entries in the above table, we claim the following:

Claim: $|a_i - b_i| = k \forall i \in \mathbf{Z}_n, k \& |a_{i+1} - b_i| = c, \forall i \mathbf{Z}_n - \{0\}, c > 0, a_i, b_i, a_{i+1} \in [0, \pi/2]$ (All a_i are equidistant from each other)

Proof: Proof by contradiction:

Assume $|a_{j+1} - b_j| = l, l \neq c, l > 0$ for some j such that maximal is achieved. If $l > c$ and as the angles are within a fixed range $[0, \pi/2]$, we would get $|a_{j+2} - b_j| < l$.
Thus, one adjacent pair to closer while the other is further. This will not satisfy the net maximality. (Contradiction).

Similar argument can be made for $l < c$ and $|a_i - b_i|$ case. ■
This can now viewed as a pictorial representation.



Thus our equation becomes,

$$\max(\min(sin^2((n-1)\alpha - n\beta), cos^2(\beta), sin^2(\alpha)))$$

with the angles being :

$$a_0 = 0, b_0 = \beta$$

$$a_i = b_0 + \beta + i(\alpha - \beta), b_i = b_0 + i(\alpha - \beta), i = 1, 2, ..., n-1$$

Given the condition that we must do better than 50%, we have

$$0 \leq \beta < \pi/4$$

$$\pi/4 < \alpha \leq \pi/2$$

$$\pi/4 < (n-1)\alpha + (2-n)\beta \leq \pi/2$$

From these conditions we get,

$$\pi/2 \leq (n-1)\alpha + (2-n)\beta < 3\pi/4$$

Some approaches to rule out here:
Consider $\alpha = \pi/2 - \beta$

The equation would reduce to:

$$\max(\min(cos^2(\beta), sin^2((n-1)\frac{\pi}{2} - (2n-1)\beta))$$

By equating cosine and sine values we get the maximum of the two:

$$\beta = \frac{(n-2)\pi}{4n}, \alpha = \frac{(n+2)\pi}{4n}$$

Thus the Winning Probability is $cos^2(\frac{(n-2)\pi}{4n}$

This cannot be the Correct Probability, because:

$$\lim_{n\to\infty} cos^2(\frac{(n-2)\pi}{4n} = 1/2$$

In theory, it has been shown for a Maximum Winning Probability of

$$\omega_q = \cos^2(\frac{\pi}{4n})$$

Chapter 4

Magic Square Game

4.1 Premise of the Game

1	1	0
0	1	1
1	1	0

Table 4.1: Example

Claim: It is not possible to have a binary 3×3 Matrix satisfying the following conditions:

1. All the rows being even parity of 1s (Even number of 1s)
2. All the columns being odd parity of 1s (Odd number of 1s)

Proof: Proof by Contradiction:
Assume there exists such a matrix M.
Let S_r & S_c be the summations of the entries taken row-wise and column-wise.
Thus,

$$S_r = \sum_{i=1}^{3} \sum_{j=1}^{3} M_{ij}$$

$$S_c = \sum_{i=1}^{3} \sum_{j=1}^{3} M_{ij}$$

Because of the parity of rows and columns,

1. S_r is even
2. S_c is odd

Summing over Matrix entries either way is the same so

$$S_r = S_c$$

However, this is a clear Contradiction, as both have different parity. Thus, $\neg\exists M \in \mathbf{R}^{3\times 3}$ ■

4.2 Description of the Game

Given an empty grid of binary 3×3 matrix.
Alice and Bob need to trick the Referee that the above theorem is false.

1. Question Sets :
   (a) $S = \{R|R \text{ is a row of } M\} \bigcup \{C|C \text{ is a column of } M\}$
   (b) $T = \mathbf{Z}_9$
2. Answer Sets :
   (a) $A = \{(a, b, c)|a, b, c \in \{0, 1\}\}$
   (b) $B = \{0, 1\}$
3. Predicate : $V(a, b|s, t)$:
   (a) a has the correct parity corresponding to s (i.e. Row with even parity and Column with odd parity)
   (b) b is equal to the corresponding entry of a

4.3 Classical Strategy

As we know there exists no such matrix satisfying the conditions, there is no classical game which can achieve a 100% Winning Probability.

Alice and Bob decide on a particular matrix.
Choice of matrix: The matrix must have a minimum no of rows/columns which violate the parity (which is 1).
WLOG, let us say that in this matrix, there is only 1 column with the wrong parity. We call it 'Column 1'.

Alice's Play for given question s: Alice is bound to answer with the correct parity, else the game is simply lost irrespective of Bob's answer

1. If s is not Column 1, she will answer honestly
2. Else, she will lie about some entries to match the column parity.

Bob's Play for given question t:
Bob will answer honestly.

Which entries will Alice lie about?
It is preferred that Alice lies about the minimum no of entries (as this decreases the chances of error concerning Bob's answer).
So she will lie about only 1 entry.

Alice and Bob know that Alice will lie about a particular entry.
So in Column 1, if Bob is asked about the non-differing entry, the answers will match.

4.3.1 Classical Winning Probability

1. # Total Questions: 18
2. # Correct Parity Rows/Columns: 5
3. # Questions from Correct Parity Rows/Columns: 15
4. In column 1, # non-differing entries : 2
5. # Questions for which they answer correctly: 17

Thus,

$$\omega_c = \frac{17}{18} = 94.45\%$$

This is the optimal classical winning probability.

4.4 Quantum Strategy

Alice would be required to fill only two bits, as once they are fixed, the third is automatically fixed according to the parity. Bob would require two bits of information as he needs to know whether Alice has been asked about a row or a column and to send the value of the entry. So, in total, we need 4 bits. Is only one entangled pair enough? No, as the no of bits required for Alice is more.

Alternate Entangled states:

1. $|GHZ\rangle = \frac{|000\rangle+|111\rangle}{\sqrt{2}}$ is not feasible as it offers only 3 bits information
2. Using 2 Entangled Pairs of state $|\psi^-\rangle$

4.4.1 Approach

We had worked on the various Entangled states, to try to get the optimal strategy, however, this could not be completed. Here is a schematic of our approach.

Our final 4 bits are:
1. Bit_0 : Alice Response
2. Bit_1 : Bob Response
3. Bit_2 : Alice Response
4. Bit_3 : Row/Column

1. We assign a unitary operator to each entry of the matrix.
2. Alice will apply the corresponding operators of a row/column to her qubits.
3. Bob will apply his entry operator on his qubit.
4. Both measure and send the answers.

We observe the following:
1. If Alice has received a Row - $Bit_3 = 0$ else $Bit_3 = 1$
2. If it is a Row, then the first 3 bits have an even parity of 1s
3. If it is a Column, then the first 3 bits have an odd parity of 1s

Thus the final desired bits should have even parity, irrespective of row/column. Thus we would want to transform our initial entangled state into one of the following states: $|0000\rangle$,$|0011\rangle$,$|0110\rangle$,$|1100\rangle$,$|0101\rangle$,$|1010\rangle$,$|1001\rangle$,$|1111\rangle$
Also, it shouldn't matter what order Alice measures the entries. So all operators of a row/column must commute.

We were not able to come up with appropriate operators for the same. In theory, There exists a perfect Quantum Strategy for the Magic Square Game.

Chapter 5

PHP Game

This is a novel Game created by us; PHP stands for Pigeonhole Principle.

This Game is based on the Pigeonhole Principle.
Alice and Bob have some $N + 1$ balls which they have to distributed into N boxes.
After distributing in some manner, they must convince the Referee that each box contains only 1 ball. The Referee will pick two balls at random and ask Alice and Bob for the respective boxes to which they belong.
Alice and Bob will win if they answer the same box for the same balls and vice versa.

Assumptions:
The referee knows only the no of balls present, not the no of boxes.
The referee knows that Alice and Bob are cheating.

According to the Pigeonhole principle : For all boxes with at least one ball, there will be one box with 2 balls.
Proof: Proof by Contradiction:
Assume that we have a mapping from $N+1$ balls to N boxes which is injective (each ball goes to only one unique box).
Let B be the set of all balls, and V be the set of all boxes. We can define a function as follows:

$$f : B \to V$$

$$f(b) = u, \forall b \in B, \exists! u \in U$$

We have $|Range(B)| = N$
But by the function equation, $|Range(B)| = N + 1$ Contradiction. ■

The Game can be formally defined as (for $n \in \mathbf{N}$):

1. Question Sets : $S = T = \mathbf{Z}_{N+1}$
2. Answer Sets : $A = B = \mathbf{Z}_N$

3. Predicate : $V(a, b|s, t) = s \oplus t \oplus a \oplus b$

Total # Questions : N^2

5.1 Reformulation

Claim: This is the same as asking: is it possible to colour a K-regular graph with only $K - 1$ colours?

Proof: We can think of each colour as a bin, which stores the vertices which have the same colour. Restricting the colours to $K - 1$ would mean that there will be two vertices with the same colour (as a K-regular graph is K-colorable, as each vertex is connected to the rest of the vertices).
Thus this is the same as the Pigeonhole Principle. ■

Thus the Game can be re-stated as follows:
Given a K-regular graph with self-loops, Alice and Bob have to trick the Referee into that it is possible to colour it with only $K - 1$ colours.

The Referee will query an edge's vertices to Alice and Bob, who would have to respond with a colour assigned to it.
If the vertices are the same, the colour has to be the same, else different.

1. Graph $G = (V, E)$
2. Vertex set $V = \mathbf{Z}_{N+1}$
3. Edge set $E = (v, w); v, w \in V$
4. Question Sets : $S = T = \mathbf{Z}_{N+1}$
5. Answer Sets : $A = B = \mathbf{Z}_N$
6. Predicate : $V(a, b|s, t) = s \oplus t \oplus a \oplus b$

Connection to the Odd Cycle Game: $K = 3$

5.2 Classical Strategy

Because of the Pigeonhole Principle, we know that the classical strategy cannot have a perfect winning strategy.

Alice and Bob will fix the configuration of the balls in the boxes.
WLOG let us say that box 1 contains two balls b_1 & b_2.
The best strategy is to simply answer honestly (as any changes will lead to more errors).

5.3 Quantum Strategy

Would it suffice to have just one entangled pair?
Notice that the outcome is of the following kind:
If the two questions are the same, Alice and Bob's answers have to match.
If the two questions are different, so do the answers.
There is only one question, so the referee would check only for this question, so he will only bother about the fact that the balls should match the box.
So each only has to answer 0/1.

Thus, they apply the same procedure as above, choosing a unitary operator. However, as there are edges for all vertices, there are too many cosines/sines terms to be satisfied, which leads to a 50% winning probability.

Thus We need more than one entangled pair.
To tackle this problem we referred to the Magic Square Game. However, as it was not resolved, this game too stands unresolved.

Chapter 6

Conclusion

In this project, we have looked at various cooperative games and studied various strategies to win them. In all of the games that we have looked at, we have made sure that the winning classical strategy is not perfect, to obtain a better quantum strategy.

We have in essence contradicted the Bell theorem's negation:
We have encoded the Non-Local hidden variable theory as Non-Local Games with cooperative players and showed that Quantum Strategies perform better than any optimal classical Strategy. These games are examples of the Bell Tests, which are a litmus test for Bell's Theorem. The study of Bell's Theorem has been done extensively, and there has been experimental verification of these results.

The reason why Quantum Strategies involving Entanglement perform better:
Entanglement provides us with two particles which are correlated. Any operation performed on one entangled qubit affects the other. Thus, the pair provides some form of communication or information relay by which the two parties can win the game. Due to this random correlation, Entanglement facilitates what is known as 'Spooky Action at a distance'.

The issue in the Classical case is once we know our question, there is no information about the second question to make any informed guesses. The best way to win in this case is to somehow gain the knowledge of the second question. In this case, we can cover up our error and increase the winning probability. This is precisely what entanglement helps us to achieve. We had also attempted to come up with our own game known as the PHP game, and to see if it also has a quantum advantage. We were able to establish the classical optimality, however, the quantum optimality was not proven.

Nevertheless, we have provided ample examples of the Quantum Advantage.

Chapter 7

References

1. Consequences and Limits of Nonlocal Strategies by Richard Cleve, Peter Høyer, Ben Toner and John Watrous

2. A simple demonstration of Bell's theorem involving two observers and no probabilities or inequalities by P.K.Aravind

3. Quantum Computation and Quantum Information 10th edition by Michael A. Nielsen and Isaac L. Chuang